ANYONS IN ELECTROMAGNETIC FIELD AND THE BMT EQUATION

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Abstract

The Lagrangian model for anyon, presented in [6], is extended to include interactions with external, homogeneous electromagnetic field. Explicit electric and magnetic moment terms for the anyon are introduced in the Lagrangian. The 2+1-dimensional BMT equation as well as the correct value (2) of the gyromagnetic ratio is rederived, in the Hamiltonian framework.

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During the last decade, 2+1-dimensional physics has been an area of intense activity. This is mainly due to the application of anyons, (2+1-dimensional particles of arbitrary spin and statistics), in realistic planar physics [1], notably fractional quantum Hall effect, high T_c - superconductivity etc.

This has provided the impetus for constructing viable classical models of free, relativistic, spinning particles in 2+1-dimensions. Starting with the work of Jackiw and Nair [2], who have derived a covariant equation of motion for free anyon, (in analogy to the Dirac equation for spin $\frac{1}{2}$ particle), a number of papers have appeared in this connection [3, 4]. However [4] does not develop a configuration space Lagrangian framework. This was attempted in [3], but unfortunately the work has given rise to criticisms [5]. On the other hand, the model proposed by the present author [6], following the authoritative work of Hanson and Regge [7], adequately describes a free anyon.

But the interesting physical properties of anyons, in particular its gyromagnetic ratio g, can only be probed in the presence of electromagnetic interactions. This was first done by Chou, Nair and Polychronakos [8],using the symplectic framework for the induced representation of the Poincare group for anyon, minimally coupled to external electromagnetic field. They found $g = 2$ for anyon, by comparing their equations with the 2+1-dimensional Bargmann-Michel-Telegdi (BMT) [9] equation. However they have not de-

rived an explicit form of the Lagrangian.

In the present Letter, we have developed a *Lagrangian* model for a 2+1-dimensional spinning particle, (anyon), with *electric and magnetic dipole moments*, interacting minimally with external electromagnetic field. Several attractive features of the present formulation and the results obtained are the following: (i) A coordinate space, reparametrization invariant Lagrangian for the interacting model is provided. Detailed constraint analysis is performed in the Hamiltonian formulation and the BMT equation is obtained via Dirac brackets [10]. Albeit classical, the BMT equation is in fact the same as the quantum mechanical Heisenberg equation of motion for the spin. We also find $g = 2$ for anyons.

(ii) For simplicity, as well as our goal of fixing the g value, we produce only $O(F_{\mu\nu})$ results, where $F_{\mu\nu}$ is the external, uniform field. All these restrictions can be lifted in a straightforward manner. A kinetic term for the gauge sector can also be included to get the full quantum theory. In fact, as we comment at the end, inclusion of the gauge kinetic term might be imperative for conceptual reasons, as has been pointed out in [7].

After this brief outline of the present work, we start with the Lagrangian

$$L = (M^2u^2 + \frac{1}{2}J^2\sigma^2 + MJ\epsilon^{\mu\nu\lambda}u_\mu\sigma_{\nu\lambda})^{\frac{1}{2}} - eu_\mu A^\mu - \frac{D}{2}\sigma_{\mu\nu}F^{\mu\nu}$$
$$= (\mathcal{L})^{\frac{1}{2}} - eu.A - \frac{D}{2}\sigma.F. \qquad (1)$$

where $\mathcal{L}$ is the same as that of the free anyon in [6] and the variables are also

defined as in [6],

$$u^\mu = \dot{x}^\mu; \quad \sigma^{\mu\nu} = \Lambda_\lambda{}^\mu \dot{\Lambda}^{\lambda\nu}; \quad \Lambda_\lambda{}^\mu \Lambda^{\lambda\nu} = \Lambda^\mu{}_\lambda \Lambda^{\nu\lambda} = g^{\mu\nu}$$

$$g^{00} = -g^{11} = -g^{22} = 1.$$

A specific (2+1-) dimensional feature of this model should be pointed out. It was shown in [7], that in 3+1-dimensions, the simple coupling term $\sigma.F$ was unphysical since it puts undue restrictions on the external field, because of the constraint structure and more complicated couplings had to be introduced. One can recognise the phenomenological electric and magnetic dipole moment terms [11] in the D-term, but constructed here out of the basic degrees of freedom.

The conjugate momenta are directly obtained as

$$P^\mu = -\frac{\partial L}{\partial u_\mu} = -\frac{1}{2}(\mathcal{L})^{\frac{-1}{2}}(2M^2 u^\mu + MJ\epsilon^{\mu\nu\lambda}\sigma_{\nu\lambda}) + eA^\mu, \tag{2}$$

$$S^{\mu\nu} = -\frac{\partial L}{\partial \sigma_{\mu\nu}} = -(\mathcal{L})^{\frac{-1}{2}}(J^2\sigma^{\mu\nu} + MJ\epsilon^{\mu\nu\lambda}u_\lambda) + DF^{\mu\nu}. \tag{3}$$

Defining $\Pi^\mu = P^\mu - eA^\mu$ and $\Sigma^{\mu\nu} = S^{\mu\nu} - DF^{\mu\nu}$, the primary (but not independent) constraints are

$$\Pi^2 = M^2; \quad \frac{1}{2}\epsilon^{\mu\nu\lambda}\Sigma_{\mu\nu}\Pi_\lambda = MJ; \quad \Sigma^2 = 2J^2, \tag{4}$$

$$V^\mu = \Sigma^{\mu\nu}\Pi_\nu = 0. \tag{5}$$

In the covariant framework, the natural choice for the independent First Class Constraints (FCC) are the mass shell condition and the Pauli-Lubanski

scalar, the first two relations of (4) respectively. This is the starting point of the Jackiw-Nair construction [2]. However, in the fixed time or Hamiltonian scheme, more useful is the choice of the mass shell condition and (5), used here, which was advocated in [7]. Note that actually (5) consists of *two* independent relations since $\Pi_\mu V^\mu = 0$ and this Second Class (SC) pair together with mass shell condition implies the Pauli-Lubanski relation. A gauge fixing for the latter in the former alternative makes the number of constraints same in both the cases. (This point was not stressed in [6].) However we are not finished with the constraints yet. The so called Weyssenhoff condition (5) ensures that in the particle rest frame, the spin has a single component S^{12}. Consequently one must introduce additional constraints that restrict the number of angular degrees of freedom to one in the rest frame as well. This is achieved by the constraint

$$\chi^\mu = \Lambda^{0\mu} - \frac{\Pi^\mu}{M} \approx 0. \tag{6}$$

The identity $\Pi^\mu \chi_\mu = -\frac{1}{2} M \chi^2$ shows that only two χ's are independent.

The fundamental Poisson Brackets (PB) are the following:

$$\{\Pi^\mu, x^\nu\} = -g^{\mu\nu}; \quad \{\Pi^\mu, \Pi^\nu\} = -eF^{\mu\nu} \tag{7}$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = S^{\mu\alpha} g^{\nu\beta} - S^{\mu\beta} g^{\nu\alpha} + S^{\nu\beta} g^{\mu\alpha} - S^{\nu\alpha} g^{\mu\beta}, \tag{8}$$

with all other PB's vanishing. Note that since $\{\Pi^\mu, \Pi^\nu\}$ PB is nontrivial, the mass shell condition in (4) has to be modified to

$$\Pi^2 - M^2 + \frac{2eF_{\mu\alpha}\Pi^\mu}{\Pi^2 - \frac{e}{2} F.\Sigma} V^\alpha \tag{9}$$

to make it an FCC. But this extra term can be ignored since we immediately move on to the Dirac Brackets (DB) generated by V^μ and hence can use $V^\mu = 0$ strongly. Let us now comput the Dirac Brackets (DB) for the SCC's V^i and χ^j for $I = 1, 2$ since only two each of the V^μ and χ^μ are independent. This manifestly non-covariant algebra can be avoided by an elegant trick, (for details see [7]) and equivalently one can construct the DB's for V^μ and χ^ν. After inverting the constraint matrix consisting of PB's of the SCC's, we get the first stage $(DB)_I$ of any two generic operators as

$$\{A, B\}_I = \{A, B\}_{PB} + \{A, V^\mu\}(\frac{eF_{\mu\nu}}{m^2M^2})\{V^\nu, B\} - \{A, \chi^\mu\}(S_{\mu\nu}$$
$$+ \frac{D}{M^2}(\Pi_\mu F_{\nu\lambda} - \Pi_\nu F_{\mu\lambda})\Pi^\lambda)\{\chi^\nu, B\} + \{A, V^\mu\}\frac{1}{M}\{\chi_\mu, B\} - \{A, \chi^\mu\}\frac{1}{M}\{V_\mu, B\} \tag{10}$$

where $m^2 = M^2 - \frac{e}{2}S.F$. Thus the $O(F)$ DB's relevant to our present interest are

$$\{\Pi^\mu, \Pi^\nu\}_I = -eF^{\mu\nu}; \quad \{\Pi^\mu, x^\nu\}_I = -g^{\mu\nu} - \frac{e}{M^2}F^{\mu\alpha}S^\nu{}_\alpha\,;$$
$$\{\Pi^\mu, S^{\nu\lambda}\}_I = \frac{e}{M^2}F^\mu{}_\alpha(\Pi^\nu S^{\alpha\lambda} - \Pi^\lambda S^{\alpha\nu}). \tag{11}$$

Note that to $O(F)$, the D terms do not appear. We are still left with the FCC $\Pi^2 - M^2$. This reflects the arbitrariness present in the definition of the parameter τ in $u^\mu = \frac{dx^\mu}{d\tau}$. We gauge fix τ to be the proper time, $x^0 - \tau \approx 0$ and now compute the final set of DB's for the pair of Scc's $\Pi^2 - M^2 \approx 0$ and $x^0 - \tau \approx 0$. The final brackets are

$$\{\Pi^\mu, S_\gamma\}^* = \frac{e}{M^2}\epsilon_{\gamma\nu\lambda}F^\mu{}_\alpha\Pi^\nu S^{\alpha\lambda} + \frac{e\epsilon_{\gamma\nu\lambda}\Pi_\alpha F^{\mu\alpha}}{\Pi^0 m^2}S^{0\nu}\Pi^\lambda, \tag{12}$$

$$\{\Pi^\mu, \Pi^\nu\}^* = -eF^{\mu\nu} + \frac{e\Pi_\alpha}{\Pi^0}(F^{\mu\alpha}g^{0\nu} + F^{\alpha\nu}g^{\mu o}) \tag{13}$$

where $S_\gamma = \frac{1}{2}\epsilon_{\gamma\mu\nu}S^{\mu\nu}$ is the relativistic generalization of the spin. Now we are equipped to derive the Hamiltonian equations of motion.

The canonical Hamiltonian H vanishes, (the theory being invariant under redefinitions of τ), but there are extra contributions to H since the gauge fixing $x^0 - \tau \approx 0$ is time dependent. This extra term is obtained in the standard way [12] and we end up with a non zero H,

$$H = \Pi^0 = (M^2 - \Pi^i\Pi_i)^{\frac{1}{2}}. \tag{14}$$

Hence the equations of motion are

$$\dot{\Pi}^\mu = \{H, \Pi^\mu\}^* = \frac{e\Pi_i}{\sqrt{(M^2 - \Pi^i\Pi_i)}}(F^{i\mu} - F^{i0}g^{0\mu}) \tag{15}$$

$$\dot{S}^\mu = \{H, S^\mu\}^* = -\frac{e}{M^3}\epsilon^{\mu\nu\lambda}\Pi_i F^{ij}\Pi_\nu S_{j\lambda}. \tag{16}$$

One can at once check that

$$\dot{\Pi}^0 = 0; \quad \dot{S}^0 = 0; \quad \dot{\Pi}^i = -\frac{e}{M}F^{ij}\Pi_j; \quad \dot{S}^i = -\frac{e}{M^3}(\Pi.S)F^{ij}\Pi_j. \tag{17}$$

Notice that $\Pi.S = MJ$ from (4) and so with the consistent choice $\frac{J}{M}\Pi^\alpha = S^\alpha$, we see that the two equations obtained separately are infact identical. We emphasize that this proof of the fact that (generalized) momentum and spin are parallel for anyons is rigorous since here the relation $\frac{J}{M}\Pi^\alpha = S^\alpha$ is an operator identity, (valid even after quantisation since we have used the Dirac Brackets), whereas stating this relation only from (4) would be purely

classical. This is equivalent to the identification made in [8]. From now on we will consider only the S^α equation, which is the appropriate BMT equation. Understandably the equations are not manifestly covariant. This is simply because we are working in the fixed-time or Hamiltonian framework. Indeed one can see that we are in the particle rest frame, (since we fixed τ to be the proper time). Hence the BMT equation (17) is at once generalized to

$$\dot{S}^\alpha = -\frac{e}{M} F^{\alpha\beta} S_\beta. \qquad (18)$$

Comparing this with the original BMT equation [8] we verify that the gyromagnatic ratio for anyons is 2.

We conclude with two comments regarding the external field formulation as well as some future directions of study. Firstly remember that we have always assumed the fields to be external, meaning that they are c-number space-time functions, devoid of a nontrivial PB structure. However, the nonzero $\{x^\mu, x^\nu\}$ DB's , originating from (10), (and present even without interactions), introduce a nonvanishing $\{A^\mu, A^\nu\}$. Does this mean that the A^μ's are no longer external? According to Hanson and Regge [7], one should include the gauge kinetic term as well from the beginning and then proceed. This will make the gauge fields dynamical variables from the start. Secondly, since the DB structure is drastically altered from the PB's, specially with respect to the $\{x^\mu, x^\nu\}$ algebra, canonical quantization is problematic. Hence one can either use directly the Wigner-Pryce [7] variables or one can try to solve the constraints perturbatively, as was done in [8].

The future prospects of this kind of Lagrangian and Hamiltonian formulations look quite promising. Since there is an explicit Lagrangian to start from, one can try to construct models for two interacting anyons. Also gravitational interactions can be introduced in a straightforward manner. Work is in progress along these directions.

References

[1] F.Wilczek, *Fractional Statistics and Anyon Superconductivity* (world Scientific, 1990)

[2] R.Jackiw and V.P.Nair, Phys.Rev.**D 43** 1933(1991)

[3] M.Chaichian, R.Gonzalez Felipe and D.Lois Martinez, Phys.Rev.Lett. **71** 3405(1993)

[4] J.-H.Cho, S.Hyun and H.-J.Lee, Phys.Lett.**B 327** 274(1994); J.-H.Cho and J.-K.Kim, Phys.Lett.**B 332** 118(1994)

[5] R.Jackiw and V.P.Nair, Comment on Phys.Rev.Lett. **71** 3405(1993), Preprint 1994; N.Banerjee, R.Banerjee and S.Ghosh, Relativistic theory of free anyon revisited, Preprint SINP/TNP/94-08

[6] S. Ghosh, Spinning particles in 2+1-dimensions, to appear in Phys.Lett. **B** 1994

[7] A.J.Hanson and T.Regge, Ann.Phys. (N.Y.), **87** 498(1974)

[8] C.Chou, V.P.Nair and A.P.Polychronakos, Phys.Lett.**B 304** 105(1993)

[9] V.Bargmann,L.Michel and V.L.Telegdi, Phys.Rev.Lett. **2** 435(1959)

[10] P.A.M.Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, 1964)

[11] A.O.Barut, *Electrodynamics of the Classical Theory of Fields and Particles*, (Macmillan Co.,N.Y. 1964)

[12] D.M.Gitman and I.V.Tyutin, *Quantization of Fields with Constraints*, (Springer-Verlag)